Tensor Part-V continued...

Inner product of two tensors: -

Let A^{ij}_{k} and B^{l}_{m} be two tensors. Consider the set of functions $A^{ij}_{k} B^{k}_{m}$ with i, j and m free indices and the index k is summed over from k = 1, 2, --- N.

There are three free indices in the function $A^{ij}_{k} B^{k}_{m}$, therefore, the number of such functions will be N^3.

In the previous lecture note we have seen than

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\rho}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\rho}}{\partial x^{l}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{l}_{m} \quad \text{--- (1)}$$

for $\rho = \gamma$, in above expression and summing over γ.

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\gamma}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{l}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{l}_{m} \quad \text{--- (2)}$$

Since $\frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{l}} = \delta^{k}_{l}$, thus we have

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\gamma}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} \delta^{k}_{l} A^{ij}_{k} B^{l}_{m}$$

using the property of delta function, we can write

$$\bar{A}^{\alpha\beta}_{\gamma}\,\bar{B}^{\gamma}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial \bar{x}^{\beta}}{\partial x^{j}}\frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{k}_{m} \quad — (3)$$

Eq. (3) shows that $A^{ij}_{k} B^{k}_{m}$ transform like the component of contravariant rank 2 and covariant rank 1. Let us define

$$\bar{C}^{\alpha\beta}_{\sigma} = \bar{A}^{\alpha\beta}_{\gamma}\,\bar{B}^{\gamma}_{\sigma} \quad — (4)$$

$$\text{and} \quad C^{ij}_{m} = A^{ij}_{k} B^{k}_{m} \quad — (5)$$

In eq (5) C^{ij}_{m} is called as the inner product of two tensors $A^{ij}_{k}\,B^{k}_{m}$.

H.W. If X^{ij}_{k} and Y^{l}_{m} are two tensors, Show that $X^{ij}_{k} Y^{i}_{m}$ is not a tensor.

Contraction of a tensor:

Let A^{ijk}_{lm} be a tensor, of rank 5, contravariant rank 2, total N^5 component. covariant rank 3,

for $l = i$, A^{ijk}_{im} will have N^3 components since index i will be summed over.

Let us write transformation equation of A^{ijk}_{lm}

$$\bar{A}^{\alpha\beta\gamma}_{\rho\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial \bar{x}^{\beta}}{\partial x^{j}}\frac{\partial \bar{x}^{\gamma}}{\partial x^{k}}\frac{\partial x^{l}}{\partial \bar{x}^{\rho}}\frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ijk}_{lm}$$

Now taking $\rho = \alpha$, and summing over α

(6)

$$\bar{A}^{\alpha\beta\gamma}_{\alpha\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \bar{x}^{\alpha}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ijk}_{lm}$$

$$= \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{k}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} \delta^{l}_{i} A^{ijk}_{lm}$$

or $$\bar{A}^{\alpha\beta\gamma}_{\alpha\sigma} = \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{k}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ijk}_{im}$$

↑

A^{ijk}_{im} → Contravariant rank 2
Covariant rank 1

When a tensor is contracted by making one of its covariant index equal to its contravariant index, then ~~the~~ resultant quantity is a tensor whose covariant and contravariant indices are reduced by one and ~~the~~ therefore the total rank is reduced by two. This process is called the contraction of a tensor.